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General solutions and scaling violation for fragmentation with mass loss

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Abstract. Explicit solutions of a linear rate equation lead to improved understanding of fragmentation with discrete and continuous mass loss. Discrete mass loss provides a general expression for the overall mass loss rate which accounts for mass loss in the shattering regime, where runaway fragmentation for small particles produces a phase of zero-mass particles. An explicit solution for the recession regime, where small particles lose large fractions of their mass to surface recession, shows that the total number of particles increases with time until the typical particles become small enough to lose all of their mass (and disappear) rather than to break. A general series solution is presented for fragmentation with continuous and discrete mass loss. A proof that continuous mass loss precludes dynamic scaling is presented.

1. Introduction

Motivated by applications such as polymer breakup, combustion, erosion, explosion and comminution (grinding), rate equations for fragmenting systems describe the time evolution of the particle mass distribution based on a particle-mass-dependent fragmentation rate and a distribution of daughter particles spawned by fragmentation. A linear rate equation describing polymer breakup due to degradation of bonds has received considerable attention (Ziff and McGrady 1985, 1986, McGrady and Ziff 1987, Cheng and Redner 1988). A nonlinear rate equation describing fragmentation due to repeated collisions between particles has also been developed (Cheng and Redner 1988). These rate equations are similar in spirit to the well-known nonlinear Smoluchowski equation for coagulation (Meesters and Ernst 1987, Mulholland and Baum 1980). Even though the spatial homogeneity inherent in rate equations is sometimes obeyed only approximately in experiments, rate equations have nevertheless added considerable insight to the overall understanding of fragmentation. Compared with numerical simulations (Sahimi and Tsotsis 1987, 1988, Kerstein and Edwards 1987), the advantage of the rate equation approach is its generality: general forms for fragmentation rates and daughter distributions allow for solutions which span a spectrum of particle morphologies, external conditions, and fragmentation processes, whereas numerical simulations typically require specific particle morphologies and external conditions.

A rate equation has recently been developed (Edwards *et al* 1990, Cai *et al* 1991) to study fragmentation for processes which do not conserve solid mass such as combustion, oxidation, dissolution and explosion. This rate equation includes both (i) *continuous* loss of solid mass (and the attendant surface recession) at particle surfaces due to the action of some continuous external process such as combustion or dissolution and (ii) *discrete* loss of mass during effectively instantaneous explosions, the mass of the exploded material accounting for the lost mass. Continuous mass loss during combustion of solid porous particles can lead to fragmentation when combustion occurs deep within the pores, thus widening the pores and eventually causing loss of connectivity of the particles. Experiments indicate that hundreds of such fragmentation events might occur during the combustion of a single coal char particle (Dunn-Rankin and Kerstein 1987, 1988, Sarofim *et al* 1977, Sundback *et al* 1985, Quann and Sarofim 1986). Discrete mass loss can lead to fragmentation during combustion of two-phase heterogeneous solids with isolated inclusions of an explosive phase imbedded within a slower-burning phase. When an explosive inclusion in a particle is exposed by surface recession, the hot gases created by the resulting explosion can break the remainder of the particle into pieces. Since continuous and discrete mass loss involve no collisions between particles and depend only on the interaction between each particle and its (assumed homogeneous) environment, the rate equation for fragmentation with mass loss is linear and reduces to the rate equation for polymer breakup in the limit of zero mass loss.

The goal of this paper is to understand implications of discrete and continuous mass loss for linear fragmentation. Our inclusion of discrete mass loss leads to a general expression for the rate of change of the overall mass of the finite-mass particles. This expression is valid even in the shattering regime, where runaway fragmentation for small particles produces a phase of zero-mass particles. A new approach using a Laplace transform yields a general family of explicit solutions for discrete mass loss, continuous values of the homogeneity index α, and arbitrary initial conditions. A new explicit solution including both continuous and discrete mass loss illuminates the competition between the reduction in the total number of particles due to particles disappearing as they lose all of their mass and the increase in the total number of particles due to fragmentation. A general series solution is obtained. A proof that continuous mass loss violates scaling strengthens previous arguments (Edwards *et al* 1990, Cai *et al* 1991).

2. Rate equation

The linear rate equation (Edwards *et al* 1990)

$$\frac{\partial n(x,t)}{\partial t} = -a(x)n(x,t) + \int_x^\infty a(y)\bar{b}(x|y)n(y,t)\,\mathrm{d}y + \frac{\partial}{\partial x}[c(x)n(x,t)] \tag{1}$$

describes the evolution of the particle mass distribution $n(x, t)$ for a system of particles undergoing fragmentation with mass loss, with a fragmentation rate $a(x)$, a distribution $\bar{b}(x|y)$ of daughter particle masses x spawned by the fragmentation of a parent particle of mass y, and a continuous mass loss rate $c(x)$. The terms on the right-hand side of (1) describe, from left to right, the reduction in the number of particles in the mass range $[x: x+\mathrm{d}x]$ due to the fragmentation of particles in this range, the increase in the number of particles in the range due to fragmentation of larger particles, and the change in the number of particles in the range due to continuous mass loss. The external consumptive process and the irrelevance of particle collisions ensure that the rate equation is linear.

A normalization condition for $\bar{b}(x|y)$,

$$y - \int_0^y x\bar{b}(x|y)\,\mathrm{d}x = \bar{\lambda} y \tag{2}$$

allows for discrete mass loss during fragmentation events; the difference between the mass y of a parent particle just before it breaks and the sum $\int_0^y x\bar{b}(x|y)\,\mathrm{d}x$ of the masses of the resulting daughter particles specifies the mass $\bar{\lambda}y$ lost during the fragmentation event, with an ensemble-averaged discrete loss fraction (Edwards *et al* 1990) satisfying $0 \leqslant \bar{\lambda} \leqslant 1$. Although $\bar{\lambda}$ may depend on y, the present work focuses on the implications of constant $\bar{\lambda}$. For two-phase heterogeneous particles, this choice corresponds to a distribution of explosive inclusions for which the average fraction $\bar{\lambda}$ of the overall particle mass contained in a particular explosive inclusion just before its ignition is independent of the particle mass. A useful general relation for the time rate of change of the total mass $M(t) = \int_0^\infty xn(x, t)\,\mathrm{d}x$ follows by multiplying (1) by x and by integrating:

$$\mathrm{d}M/\mathrm{d}t = -\int_0^\infty [x\bar{\lambda}a(x) + c(x)]n(x, t)\,\mathrm{d}x. \tag{3}$$

As will be seen, this expression accounts for the overall mass of the *finite-mass* particles. The familiar rate equation for (mass conserving) polymer degradation (McGrady and Ziff 1987, Cheng and Redner 1988) follows from (1) by setting $c(x) = 0$. The corresponding normalization condition follows from (2) with $\bar{\lambda} = 0$.

Using dimensionless variables, we consider the power-law rates $a(x) = x^\alpha$, $\bar{b}(x|y) = g(y)x^\nu$, and $c(x) = \epsilon x^\gamma$ with $\epsilon \geqslant 0$. As discussed by Edwards *et al* (1990) and Cai *et al* (1991), these power-law rates describe a wide spectrum of conditions for different values of α, γ, ν and ϵ. In particular, the conditions $\sigma = \gamma - \alpha - 1 < 0$, $\sigma = 0$ and $\sigma > 0$ yield the recession, scaling and fragmentation regimes: in the recession regime, small particles lose large fractions of their masses to surface recession between fragmentation events and eventually disappear. In the fragmentation regime, small particles lose very little mass between fragmentation events and continue to break into smaller and smaller particles. The scaling regime, where the rate equation is scale invariant, is the boundary between these two regimes. The coefficient ϵ measures the importance of mass loss relative to fragmentation. The normalization condition (equation (2)) implies that $g(y) = 2\phi/y^{\nu+1}$ with $\phi = (1-\bar{\lambda})(\nu+2)/2$ and $\nu > -2$, so that (1) becomes

$$\frac{\partial n(x, t)}{\partial t} = -x^\alpha n(x, t) + 2\phi x^\nu \int_x^\infty y^{\alpha-\nu-1} n(y, t)\,\mathrm{d}y + \epsilon \frac{\partial}{\partial x}[x^\gamma n(x, t)]. \tag{4}$$

The ensemble-averaged number of daughter particles produced by a fragmentation event is given by $\bar{\mathcal{N}} = \int_0^y \bar{b}(x|y)\,\mathrm{d}x = 2\phi/(\nu+1)$ for $\nu > -1$, whereas $\bar{\mathcal{N}} = \infty$ for $-2 < \nu \leqslant -1$ (McGrady and Ziff 1987). The requirement that fragmentation events produce two or more fragments, $\bar{\mathcal{N}} \geqslant 2$, immediately yields the upper limits of the allowed ranges $-1 < \nu \leqslant -2\bar{\lambda}/(1+\bar{\lambda})$ and $(1-\bar{\lambda})/2 < \phi \leqslant (1-\bar{\lambda})/(1+\bar{\lambda})$, whereas $\bar{\mathcal{N}} < \infty$ implies the lower limits.

It is helpful to transform according to $x = u^{1/\phi}$ and $n(x, t) = x^\nu w(u, t)$, so that

$$\frac{\partial w(u, t)}{\partial t} = -u^\beta w(u, t) + 2\int_u^\infty v^{\beta-1} w(v, t)\,\mathrm{d}v + (\delta+\mu)\eta u^{\delta-1} w(u, t) + \eta u^\delta \frac{\partial w(u, t)}{\partial u} \tag{5}$$

where $\beta = \alpha/\phi$, $\delta = 1 + (\gamma-1)/\phi$, $\eta = \epsilon\phi$ and $\mu = [\nu + \bar{\lambda}(\nu+2)]/2\phi$ satisfies $-1 < \mu \leqslant 0$.

3. Explicit solutions for discrete mass loss

To obtain a general family of explicit solutions in the absence of continuous mass loss, we set $\epsilon = 0$ in (5):

$$\frac{\partial w(u,t)}{\partial t} = -u^{\beta} w(u,t) + 2\int_u^{\infty} v^{\beta-1} w(v,t)\,\mathrm{d}v. \tag{6}$$

Compared with the rate equation for polymer degradation (McGrady and Ziff 1987), the essential new feature of this equation is a non-zero discrete mass loss fraction $\bar{\lambda}$ implicit in $\beta = 2(1-\bar{\lambda})^{-1}(\nu+2)^{-1}\alpha$, as required by the normalization condition (2). Our direct-solution method involving a Laplace transform, which is distinct from the trial-solution method employed by Ziff and McGrady (1985), allows us to obtain solutions for the continuous spectrum of values, $-\infty < \alpha < \infty$, and for arbitrary initial conditions. Our method is also easily generalized for other time-independent forms of $a(x)$ and $b(x|y)$, and may prove useful in obtaining solutions including continuous mass loss.

In appendix A, we use the Laplace transform method to solve (6). For $m = (1-\bar{\lambda})(\nu+2)/\alpha > 0$, $\epsilon = 0$, and arbitrary initial conditions, the resulting general solution of (4) is

$$n(x,t) = \mathrm{e}^{-x^{\alpha}t}\left[n(x,0) + mtx^{\nu}\int_x^{\infty} y^{-\nu} n(y,0) F_1(1-m, 2, t(x^{\alpha}-y^{\alpha}))\,\mathrm{d}y^{\alpha}\right] \tag{7}$$

where $F_1(a, b, x)$ is the confluent hypergeometric function. When m is a positive integer, (7) reduces to

$$n(x,t) = \mathrm{e}^{-x^{\alpha}t}\left[n(x,0) + tx^{\nu}\int_x^{\infty} y^{-\nu} n(y,0) L_{m-1}^{(1)}(t(x^{\alpha}-y^{\alpha}))\,\mathrm{d}y^{\alpha}\right] \tag{8}$$

where

$$L_{m-1}^{(1)}(x) = \sum_{j=1}^{m} \frac{m!}{(m-j)!j!}\frac{(-x)^{j-1}}{(j-1)!}$$

is the associated Laguerre polynomial. For $\nu = \bar{\lambda} = 0$ corresponding to binary breakup with $\bar{\mathcal{N}} = 2$, (8) reduces to the result obtained by Ziff and McGrady (1985).

For the simple case $m = 1$ $[\alpha = (1-\bar{\lambda})(\nu+2)]$ with a monodisperse initial distribution $n(x,0) = \delta(x-l)$, (8) yields

$$n(x,t) = \mathrm{e}^{-x^{\alpha}t}[\delta(x-l) + \alpha t l^{\alpha-\nu-1} x^{\nu}] \tag{9}$$

for $x \leqslant l + \varepsilon$ with $\varepsilon \to 0$, and $n(x,t) = 0$ otherwise. The corresponding total mass is

$$M(t) = l\,\mathrm{e}^{-l^{\alpha}t} + l^{\alpha-\nu-1} t^{-\bar{\lambda}/(1-\bar{\lambda})}\int_0^{l^{\alpha}t} \mathrm{e}^{-z} z^{\bar{\lambda}/(1-\bar{\lambda})}\,\mathrm{d}z. \tag{10}$$

As $t \to \infty$, we have

$$M(t) \approx l^{1-\bar{\lambda}(\nu+2)} t^{-\bar{\lambda}/(1-\bar{\lambda})}\Gamma\left(\frac{1}{1-\bar{\lambda}}\right). \tag{11}$$

Hence, discrete mass loss yields a total mass which decreases as a power of t as $t \to \infty$, with an exponent determined by the discrete mass loss fraction $\bar{\lambda}$. When $\bar{\lambda} = 0$, both (10) and (11) reduce to the result for mass-conserving fragmentation, $M(t) = l$.

By using Kummer's identity $F_1(a, b, x) = e^x F_1(b-a, b, -x)$ (Arfken 1985, section 13.6), we can extend the general solution (7) to the case $m<0$ and $\epsilon=0$;

$$n(x,t) = e^{-x^\alpha t} n(x,0) + mtx^\nu \int_x^\infty e^{-y^\alpha t} y^{-\nu} n(y,0) F_1(1+m, 2, t(y^\alpha - x^\alpha))\, dy^\alpha. \tag{12}$$

When m is a negative integer, (12) reduces to

$$n(x,t) = e^{-x^\alpha t} n(x,0) - tx^\nu \int_x^\infty e^{-y^\alpha t} y^{-\nu} n(y,0) L^{(1)}_{-m-1}(t(y^\alpha - x^\alpha))\, dy^\alpha. \tag{13}$$

For $m=-2$ $(\alpha = -(1-\bar\lambda)(\nu+2)/2 < 0)$ and the monodisperse initial distribution $n(x,0) = \delta(x-l)$, (13) reduces to

$$n(x,t) = \exp(-tl^\alpha)\{\delta(x-l) + (1-\bar\lambda)(\nu+2)tx^\nu l^{\alpha-\nu-1}[1-(l^\alpha - x^\alpha)t/2]\} \tag{14}$$

for $x \leqslant l$. This distribution yields a total initial number of particles $N(0) = \int_0^\infty n(x,0)\, dx = 1$ consistent with monodisperse initial conditions. The condition $-(1-\bar\lambda)/2 < \alpha+\nu+1 \leqslant 0$ (which follows from the allowable range for ν) requires a diverging total number of particles

$$N(t) = \int_0^\infty n(x,t)\, dx = \frac{\alpha l^{\alpha-\nu-1}}{\alpha+\nu+1} t^2 e^{-tl^\alpha} x^{\alpha+\nu+1}|_{x\to 0} + N_f(t) \tag{15}$$

for $t>0$ with a finite term $N_f(t)$. The corresponding total mass of the finite-mass particles

$$M(t) = l\exp(-tl^\alpha)\left[1 + (1-\bar\lambda)tl^\alpha + \frac{(1-\bar\lambda)^2}{1+\bar\lambda}\frac{(tl^\alpha)^2}{2}\right] \tag{16}$$

decreases monotonically with time. These results reflect the runaway fragmentation rates $a(x) = x^\alpha$ $(\alpha<0)$ for small particles associated with the 'shattering' regime, where the total mass $M(t)$ of the finite-mass particles decreases with time as finite-mass particles are reduced to infinite numbers $N(t)$ of infinitesimal-mass particles. In the limit $\bar\lambda \to 0$, the total mass

$$M(t) = l\exp(-tl^{-1-\nu/2})[1 + tl^{-1-\nu/2} + \tfrac{1}{2}t^2 l^{-\nu-2}] \tag{17}$$

agrees with the corresponding result obrained by McGrady and Ziff (1987) and the total number of particles

$$N(t) = -\frac{\nu+2}{\nu} l^{-2-3\nu/2} t^2 e^{-tl^{-1-\nu/2}} x^{\nu/2}|_{x\to 0} + N_f(t) \tag{18}$$

can be obtained by direct integration of their $n(x,t)$. Note that whereas (16) reflects both discrete mass loss (associated with non-zero $\bar\lambda$) and shattering mass loss (associated with the reduction of finite-mass particles to infinite numbers of infinitesimal-mass particles), (17) reflects only shattering mass loss, the quantity $l - M(t)$ giving the amount of mass embodied in the infinitesimal-mass particles.

Including discrete mass loss accounts for all mass loss including shattering mass loss in (3), $dM/dt = -\bar\lambda \int_0^\infty x^{\alpha+1} n(x,t)\, dx$; integrating this equation using (14) yields a finite mass loss rate

$$dM/dt = \left[-\bar\lambda - 2\bar\lambda\frac{1-\bar\lambda}{1+\bar\lambda} l^\alpha t - \frac{1}{2}\frac{(1-\bar\lambda)^2}{1+\bar\lambda} l^{2\alpha} t^2\right] e^{-tl^\alpha} l^{1+\alpha}$$

which is consistent with the time derivative of (16). Equation (3) is valid even in the limit $\bar{\lambda}\to 0$, where shattering requires the integral

$$\int_0^\infty x^{\alpha+1} n(x,t)\,\mathrm{d}x \to \frac{1}{2}\bar{\lambda}^{-1} t^2 l^{-2-3\nu/2} \exp(-tl^{-1-\nu/2})$$

to diverge as $\bar{\lambda}^{-1}$ in order to produce a finite mass loss rate dM/dt which agrees with (17). Thus, introducing discrete mass loss renders the appropriate integrals convergent and thereby accounts for shattering mass loss in a natural way; the limit $\bar{\lambda}\to 0$ is properly taken at the end of the calculation. Discrete models of fragmentation in which particle masses are restricted to multiples of a smallest mass Δ can also be used to account for shattering mass loss in the limit $\Delta\to 0$ (Ziff and McGrady 1986). Equation (3) can be further confirmed by considering (13) with monodisperse initial conditions.

4. Recession regime behaviour

In appendix B, we obtain a series solution of (5) including both discrete and continuous mass loss. The coefficients of the series are obtained recursively based on the initial condition $w(u, 0)$.

The special case $\alpha = 1$, $\gamma = 0$, $\bar{\lambda} = 0$ and $\nu = 0$ illustrates the behaviour in the recession regime, where negative $\sigma = \gamma - \alpha - 1 = -2$ implies that small particles typically lose all of their mass without breaking. The recession regime is relevant when pore sizes have a minimum and for high temperatures and/or low oxygen concentrations relevant to diffusion-limited oxidation. For this special case, (B4) implies that the x dependence of all coefficients of the series is e^{-sx}. As a result, $n_s(x,t)$ must have the form $n_s(x,t) = e^{-(t+s)x} f_s(t)$, which we substitute into (1) to obtain

$$n_s(x,t) = (1+t/s)^2 \exp[-(s+t)x - 2^{-1}\epsilon(t+s)^2]. \tag{19}$$

Because of the linearity of the rate equation, the general solution is

$$n(x,t) = \int_0^\infty A(s)(1+t/s)^2 \exp[-(s+t)x - 2^{-1}\epsilon(t+s)^2]\,\mathrm{d}s. \tag{20}$$

Expanding $(1+t/s)^2$ and using the identities

$$n(x,0) = \int_0^\infty A(s)\, e^{-2^{-1}\epsilon s^2} e^{-sx}\,\mathrm{d}s$$

and

$$\int_x^\infty (y-x)^k n(y,0)\,\mathrm{d}y = k! \int_0^\infty s^{-k-1} A(s)\, e^{-2^{-1}\epsilon s^2} e^{-sx}\,\mathrm{d}s$$

yields an explicit solution

$$n(x,t) = e^{2^{-1}\epsilon t^2 - \xi t}\left\{ n(\xi,0) + \int_\xi^\infty n(y,0)[2t + t^2(y-\xi)]\,\mathrm{d}y \right\} \tag{21}$$

governing the evolution of the particle mass distribution for an arbitrary initial distribution, with $\xi = x + \epsilon t$. When $\epsilon = 0$, (21) reduces to the expression for mass-conserved fragmentation given by Ziff and McGrady (1985). Equation (21) illustrates that continuous mass loss is relevant at large t (through the factor $e^{2^{-1}\epsilon t^2}$) when the particles are typically small.

As an explicit illustrative example, we consider the initial distribution $n(x,0)=2(x+k)^{-3}$, for which (21) gives

$$n(x,t)=\left[\frac{t^2}{\epsilon t+x+k}+\frac{2t}{(\epsilon t+x+k)^2}+\frac{2}{(\epsilon t+x+k)^3}\right]\exp[-2^{-1}\epsilon t^2-tx]. \tag{22}$$

The evolution of the total mass and the total number of particles in the system are given by

$$M(t)=\frac{e^{-2^{-1}\epsilon t^2}}{k+\epsilon t} \tag{23}$$

and

$$N(t)=\left[\frac{1}{(k+\epsilon t)^2}+\frac{t}{k+\epsilon t}\right]e^{-2^{-1}\epsilon t^2}. \tag{24}$$

Thus, $k=1/M_0$, where M_0 is the total initial mass of the system. Without mass loss ($\epsilon=0$), the total mass $M(t)=M_0$ is constant, the total number $N(t)=N_0+t/k$ is a linearly increasing function of time, and $n(x,t)\approx t^2\,e^{-tx}$ is a scaling solution at long times when the particle masses are small compared with k. With mass loss ($\epsilon\neq 0$), the total mass of the system is a monotonically decreasing function of time, as expected, whereas the evolution of the total number of particles depends on the mass loss rate ϵ. For large mass loss rates $\epsilon\geqslant k^2/2$, the total number decreases monotonically with time. For small mass loss rates $\epsilon<k^2/2$, the total number increases up to a critical time t_c, after which it decreases monotonically with time. The crossover from increasing to decreasing total number of particles for small loss rates signals the time at which typical particles become small enough to lose all of their mass rather than to break. Although this behaviour for small and large mass loss rates has been shown rigorously only for this special case, it is expected that general initial distributions and exponent values in the recession regime ($\sigma<0$) will also reflect this behaviour.

5. Scaling violation with continuous mass loss

Scale-invariant 'scaling' solutions for coagulation hold interest because of evidence (Meesters and Ernst 1987, Mulholland and Baum 1980, Family *et al* 1986) that large classes of general solutions tend to scaling solutions after initial transients decay away. For polymer breakup, the scaling solution has been studied by Cheng and Redner (1988). For fragmentation with discrete mass loss, exact and asymptotic scaling solutions are given in Cai *et al* 1991. In Edwards *et al* (1990), analysis of the moment equations leads to an argument that continuous mass loss violates scaling. Here, we present a stronger argument that continuous mass loss violates scaling based on the original differential equation.

Invariance of (5) under the scale transformation $u=su^*$, $t=s^{\varphi}t^*$ and $w(u,t)=w(su^*,s^{\varphi}t^*)=s^{\psi}w^*(u^*,t^*)$ requires $\delta=\beta+1$, $\varphi=-\beta$, and a form for scaling solutions,

$$w(u,t)=t^{-\psi/\beta}f(u^{\beta}t). \tag{25}$$

Letting $\xi=\beta\eta u^{\beta}t$, substituting (25), and differentiating reduces (5) to the hypergeometric equation

$$\xi(\xi-1)f''(\xi)+[(1+a+b)\xi-c]f'(\xi)+abf(\xi)=0 \tag{26}$$

with

$$a = \frac{1}{2\beta\eta}\{(2\beta\eta + \eta + \mu\eta - 1) + [1 + (6 - 2\mu)\eta + (1+\mu)^2\eta^2]^{1/2}\}$$
$$b = \frac{1}{2\beta\eta}\{(2\beta\eta + \eta + \mu\eta - 1) - [1 + (6 - 2\mu)\eta + (1+\mu)^2\eta^2]^{1/2}\} \tag{27}$$
$$c = 1 - \psi/\beta.$$

For large ξ, the two independent solutions of (26) are

$$\begin{aligned} f_1(\xi) &= \xi^{-a}F(a, a-c+1, a-b+1, \xi^{-1}) \\ f_2(\xi) &= \xi^{-b}F(b, b-c+1, b-a+1, \xi^{-1}) \end{aligned} \tag{28}$$

where $F(a, b, c, x)$ is the hypergeometric function. Substituting the total mass

$$M(t) = \frac{2}{(\nu+2)}(\beta\eta)^{-2/\beta}t^{-(\psi+2)/\beta}\int_0^\infty \xi^{1/\beta}f(\xi)\,\mathrm{d}\xi^{1/\beta} \tag{29}$$

into (3), we have

$$(\psi+2)\int_0^\infty \xi^{1/\beta}f(\xi)\,\mathrm{d}\xi^{1/\beta} = \frac{2}{(\nu+2)}\int_0^\infty \xi^{(\beta+1)/\beta}f(\xi)\,\mathrm{d}\xi^{1/\beta} \tag{30}$$

which determines the value of ψ.

As $\xi \to \infty$, we have $f_1(\xi) \to \xi^{-a}$ and $f_2(\xi) \to \xi^{-b}$, so that the convergence of

$$\int_0^\infty \xi^{(\beta+1)/\beta}f(\xi)\,\mathrm{d}\xi^{1/\beta}$$

requires

$$a > 2/\beta + 1 \qquad b > 2/\beta + 1 \tag{31}$$

respectively for the two independent solutions. Substituting (27) into (31) yields the inequalities

$$\begin{aligned} &[1 + (6-2\mu)\eta + (1+\mu)^2\eta^2]^{1/2} > 1 + (3-\mu)\eta \\ &-[1 + (6-2\mu)\eta + (1+\mu)^2\eta^2]^{1/2} > 1 + (3-\mu)\eta. \end{aligned} \tag{32}$$

The allowed ranges $-1 < \mu \leqslant 0$ and $\eta > 0$ disagree with the second inequality in (32). Squaring the first inequality implies that $\mu > 1$, which also disagrees with the allowed range of μ. Thus, the scaling solutions $f_1(\xi)$ and $f_2(\xi)$ cannot be physical solutions of (26), implying that continuous mass loss violates scaling.

6. Conclusions

Explicit and series solutions help to illuminate the physics of fragmentation with mass loss. Discrete mass loss allows for a general relation for the mass loss rate which includes shattering mass loss in a natural way, thereby improving the understanding of shattering mass loss. Shown to be useful for power-law forms of the fragmentation rate $a(x)$ and daughter distribution $b(x|y)$, our general solution method using a Laplace transform may be useful for other time-independent forms of these kernels. An explicit solution illustrates the general behaviour in the recession regime, where small particles lose large fractions of their mass to surface recession before fragmentation or annihilation.

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Appendix A. Laplace transform solution

By applying the Laplace transform

$$W(u,s)=\int_0^\infty e^{-st}w(u,t)\,dt \tag{A1}$$

and by differentiating (6) with respect to u, we obtain

$$(s+u^\beta)\frac{\partial W(u,s)}{\partial u}+(\beta+2)u^{\beta-1}W(u,s)=\frac{dw(u,0)}{du}. \tag{A2}$$

The solution of this first-order linear inhomogeneous differential equation can be written down immediately (see Arfken 1985, section 8.2, for example):

$$W(u,s)=\frac{w(u,0)}{s+u^\beta}+m\int_u^\infty w(v,0)\frac{(s+v^\beta)^{m-1}}{(s+u^\beta)^{(1+m)}}\,dv^\beta+g(s,u) \tag{A3}$$

where $g(s,u)=C(s)/(s+u^\beta)^{(1+m)}$, $C(s)$ is constant of integration depending on s, and $m=2/\beta$.

For $m>0$, we can employ

$$(a+b)^n=\sum_{k=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-k)}\frac{a^{n-k}b^k}{k!} \tag{A4}$$

and

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z} \tag{A5}$$

to reduce (A3) to

$$W(u,s)=\frac{w(u,0)}{s+u^\beta}+m\int_u^\infty w(v,0)\sum_{k=0}^{\infty}\frac{\Gamma(1-m+k)}{\Gamma(1-m)}\frac{(u^\beta-v^\beta)^k}{k!(s+u^\beta)^{k+2}}\,dv^\beta+g(s,u) \tag{A6}$$

where $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt$ is the gamma function. The inverse Laplace transform gives

$$w(u,t)=e^{-u^\beta t}\left[w(u,0)+mt\int_u^\infty w(v,0)F_1(1-m,2,t(u^\beta-v^\beta))\,dv^\beta\right]$$
$$+\mathscr{L}^{-1}(g(s,u)). \tag{A7}$$

where

$$F_1(a,b,x)=\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(a)}\frac{\Gamma(b)}{\Gamma(b+k)}\frac{x^k}{k!}$$

is the confluent hypergeometric function and $\mathscr{L}^{-1}$ represents the inverse Laplace transform.

The quantity $\mathscr{L}^{-1}(g(s, u))$ is independent of the form of the initial distribution $w(u, 0)$. For the initial distribution $w(u, 0)=0$, the solution $w(u, t)=0$ requires that $\mathscr{L}^{-1}(g(s, u))=0$, hence $\mathscr{L}^{-1}(g(s, u))=0$ generally. Thus, for $m=(1-\bar{\lambda})(\nu+2)/\alpha>0$, $\epsilon=0$ and arbitrary initial conditions, the general solution of (4) is

$$n(x, t)=\mathrm{e}^{-x^{\alpha}t}\left[n(x, 0)+mtx^{\nu}\int_{x}^{\infty} y^{-\nu}n(y, 0)F_1(1-m, 2, t(x^{\alpha}-y^{\alpha}))\,\mathrm{d}y^{\alpha}\right]. \tag{A8}$$

Appendix B. General series solution

The solution of (6) has the form

$$w(u, t)=\mathrm{e}^{-u^{\beta}t}f(u, t) \tag{B1}$$

for $m>0$ and discrete mass loss only ($\epsilon=0$), with $f(u, t)$ specified by (A7). We now use (B1) as an assumed form for a general series solution which includes continuous mass loss. Substituting (B1) into (5) and differentiating with respect to u, we obtain

$$\begin{aligned}\frac{\partial^2 f}{\partial u\partial t}&+\left(2-\beta t\frac{\partial}{\partial t}\right)u^{\beta-1}f\\ &=\eta\left[(\delta+\mu)\frac{\partial}{\partial u}(u^{\delta-1}f)+\frac{\partial}{\partial u}\left(u^{\delta}\frac{\partial f}{\partial u}\right)-\beta(2\delta+\beta+\mu-1)tu^{\delta+\beta-2}f\right.\\ &\qquad\left.-2\beta tu^{\delta+\beta-1}\frac{\partial f}{\partial u}+\beta^2t^2u^{\delta+2\beta-2}f\right].\end{aligned} \tag{B2}$$

For the initial condition $w(u, 0)=f(u, 0)=\mathrm{e}^{-su^{\beta}}$ with $s\geqslant 0$, (B2) can be solved by substituting the series expansion

$$f_s(u, t)=\sum_{k=0}^{\infty} A_k(u, s)\frac{t^k}{k!} \tag{B3}$$

equating like powers of t, and integrating over u. This procedure yields the recurrence relation

$$\begin{aligned}A_{k+1}(u, s)=(2-k\beta)\int_u^{\infty} u^{\beta-1}A_k(v, s)\,\mathrm{d}v+\eta\Bigg[&-2k\beta u^{\delta+\beta-1}A_{k-1}(u, s)\\ &+(\delta+\mu)u^{\delta-1}A_k(u, s)-k\beta(\beta-\mu-1)\int_u^{\infty} v^{\delta+\beta-2}A_{k-1}(v, s)\,\mathrm{d}v\\ &+u^{\delta}\frac{\mathrm{d}A_k(u, s)}{\mathrm{d}u}-k(k-1)\beta^2\int_u^{\infty} v^{\delta+2\beta-2}A_{k-2}(v, s)\,\mathrm{d}v\Bigg].\end{aligned} \tag{B4}$$

Because of the linearity of (5), the corresponding general series solution

$$w(u, t)=\mathrm{e}^{-u^{\beta}t}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^{\infty} B(s)A_k(u, s)\,\mathrm{d}s \tag{B5}$$

reflects arbitrary initial conditions through $B(s)$ which is determined by

$$B(s)=\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} w(u, 0)\,\mathrm{e}^{su^{\beta}}\,\mathrm{d}u^{\beta}. \tag{B6}$$

Thus, given arbitrary initial conditions $w(u, 0)$, (B5) gives the general solution for $w(u, t)$, with $B(s)$ given by (B6) and with recurrence relations for $A_k(u, s)$ given by (B4).

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